

**ON THE KONTSEVICH AND THE
CAMPBELL–BAKER–HAUSDORFF DEFORMATION
QUANTIZATIONS OF A LINEAR POISSON STRUCTURE**

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ABSTRACT. For the Kirillov–Poisson structure on the vector space \mathfrak{g}^* , where \mathfrak{g} is a finite-dimensional Lie algebra, it is known at least two canonical deformations quantization of this structure: they are the M. Kontsevich universal formula [K], and the formula, arising from the classical Campbell–Baker–Hausdorff formula [Ka]. It was proved in [Ka] that the last formula is exactly the part of Kontsevich’s formula consisting of all the admissible graphs without (oriented) cycles between the vertices of the first type. It follows from the CBH-theorem that this part of Kontsevich’s formula defines an *associative* product (in the case of a linear Poisson structure).

The aim of these notes is to prove the last result directly, using the methods analogous to [K] instead of the CBH-formula. We construct an L_∞ -morphism $\mathcal{U}_{\text{lin}}: [T_{\text{poly}}^\bullet]_{\text{lin}} \rightarrow D_{\text{poly}}^\bullet$ from the dg Lie algebra of polyvector fields with *linear coefficients* to the dg Lie algebra of polydifferential operators, which is *not* equal to the restriction of the Formality L_∞ -morphism $\mathcal{U}: T_{\text{poly}}^\bullet \rightarrow D_{\text{poly}}^\bullet$ [K] to the subalgebra $[T_{\text{poly}}^\bullet]_{\text{lin}}$. For a bivector field α with linear coefficients such that $[\alpha, \alpha] = 0$ the corresponding solution $\mathcal{U}_{\text{lin}}(\alpha)$ of the Maurer–Cartan equation in D_{poly}^\bullet defines exactly the CBH-quantization, in the case of the harmonic angle map [K], Sect.2. We prove the associativity of the restricted Kontsevich formula (in the linear case) also for any angle map [K], Sect.6.2.

1. L_∞ -MORPHISMS, THE MAURER–CARTAN EQUATION, AND *-PRODUCTS

Let $\mathcal{F}: T_{\text{poly}}^\bullet \rightarrow D_{\text{poly}}^\bullet$ be an L_∞ -morphism from the dg Lie algebra of polyvector fields on \mathbb{R}^d to the dg Lie algebra of polydifferential operators on \mathbb{R}^d , and let

$$\begin{aligned}\mathcal{F}_1 &: T_{\text{poly}}^\bullet \rightarrow D_{\text{poly}}^\bullet \\ \mathcal{F}_2 &: \wedge^2 T_{\text{poly}}^\bullet \rightarrow D_{\text{poly}}^\bullet[-1] \\ \mathcal{F}_3 &: \wedge^3 T_{\text{poly}}^\bullet \rightarrow D_{\text{poly}}^\bullet[-2] \\ &\dots\end{aligned}$$

be its Taylor components.

Than any solution $\alpha \in T_{\text{poly}}^1$ of the Maurer–Cartan equation (i.e. α is a bivector field such that $[\alpha, \alpha] = 0$) defines a solution $\mathcal{F}(\alpha) \in D_{\text{poly}}^1$ of the Maurer–Cartan equation in D_{poly}^\bullet ($\mathcal{F}(\alpha) \in \text{Hom}_{\mathbb{C}}(C^\infty(\mathbb{R}^d)^{\otimes 2} \rightarrow C^\infty(\mathbb{R}^d))$) as follows:

$$(1) \quad \mathcal{F}(\alpha) = \mathcal{F}_1(\alpha) + \frac{1}{2}\mathcal{F}_2(\alpha, \alpha) + \frac{1}{6}\mathcal{F}_3(\alpha, \alpha, \alpha) + \dots + \frac{1}{n!}\mathcal{F}_n(\alpha, \dots, \alpha) + \dots$$

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One can prove that the bidifferential operator $\mathcal{F}(\alpha)$ satisfy the Maurer–Cartan equation

$$(2) \quad d\mathcal{F}(\alpha) + \frac{1}{2}[\mathcal{F}(\alpha), \mathcal{F}(\alpha)] = 0$$

where d is the Hochschild differential and $[,]$ is the Gerstenhaber bracket. It follows directly from the definitions that (2) is equivalent to the statement that the formula

$$(3) \quad f * g = f \cdot g + \mathcal{F}(\alpha)(f \otimes g)$$

defines an *associative* product on the vector space $C^\infty(\mathbb{R}^d)$.

2. FORMALITY L_∞ -MORPHISM $\mathcal{U}: T_{\text{poly}}^\bullet \rightarrow D_{\text{poly}}^\bullet$ [K]

2.1. Admissible Graphs and Weights.

Definition. Admissible graph is an oriented graph with labels such that

- 1) the set of vertices V_Γ is $\{1, \dots, n\} \coprod \{\bar{1}, \dots, \bar{m}\}$ where $n, m \in \mathbb{Z}_{\geq 0}$; vertices from the set $\{1, \dots, n\}$ are called vertices of the first type, vertices from $\{\bar{1}, \dots, \bar{m}\}$ are called vertices of the second type,
- 2) every edge $(v_1, v_2) \in E_\Gamma$ starts at a vertex of first type, $v_1 \in \{1, \dots, n\}$;
- 3) there are no loops, i.e. no edges of the type (v, v) ;
- 4) for every vertex $k \in \{1, \dots, n\}$ of the first type, the set of edges

$$\text{Star}(k) := \{(v_1, v_2) \in E_\Gamma \mid v_1 = k\}.$$

starting from s is labeled by symbols $(e_k^1, \dots, e_k^{\text{Star}(k)})$.

For any admissible graph Γ , we define weight $W_\Gamma \in \mathbb{C}$ by formula

$$W_\Gamma = \prod_{k=1}^n \frac{1}{(\#\text{Star}(k))!} \cdot \frac{1}{(2\pi)^{2n+m-2}} \int_{C_{n,m}^+} \bigwedge_{e \in E_\Gamma} d\varphi_e.$$

Let us explain written here. Let

$$\text{Conf}_{n,m} = \{(p_1, \dots, p_n; q_1, \dots, q_m) \mid p_i \in \mathcal{H}, q_j \in \mathbb{R}, p_{i_1} \neq p_{i_2} \text{ for } i_1 \neq i_2 \text{ and } q_{j_1} \neq q_{j_2} \text{ for } j_1 \neq j_2\}.$$

Here $\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. Let G be a group of affine transformations $G = \{z \mapsto az + b, a, b \in \mathbb{R}, a > 0\}$.

Then $\text{Conf}_{n,m}^+ = \{p_1, \dots, p_n; q_1, \dots, q_m\} \in \text{Conf}_{n,m} \mid q_1 < q_2 < \dots < q_m\}$ is invariant under the action of G , and we define $C_{n,m} = \text{Conf}_{n,m}/G$, $C_{n,m}^+ = \text{Conf}_{n,m}^+/G$.

Every edge $e \in E_\Gamma$ defines a map from $\text{Conf}_{n,m}$ to $\text{Conf}_{2,0}$ (if two end-points of e are of the first type) and to $\text{Conf}_{1,1}$ otherwise. For $p, q \in \mathcal{H} \sqcup \mathbb{R}$ ($p \neq q$) we define function

$$\Phi(p, q) = \text{Arg} \left(\frac{(q-p)}{(q-\bar{p})} \right) = \frac{1}{2i} \text{Log} \left(\frac{(q-p)(\bar{q}-p)}{(q-\bar{p})(\bar{q}-\bar{p})} \right)$$

and 1-form $d\Phi$.

This function is G -invariant, and this construction defines a 1-form $d\Phi_e$ for any $e \in E_\Gamma$, which is the pull-back of $d\Phi$.

Lemma. *Integral in the definition of W_Γ is absolutely convergent for any Γ .*

The proof is done in Section 5 of [K]. □

2.2. Formality Morphism. For any admissible graph Γ with n vertices of the first type, m vertices of the second type, and $2n + m - 2 + l$ edges where $l \in \mathbb{Z}$, we define a linear map $\mathcal{U}_\Gamma : \otimes^n T_{poly}(\mathbb{R}^d) \rightarrow D_{poly}(\mathbb{R}^d)[1 + l - n]$. This map has only one non-zero graded component $(\mathcal{U}_\Gamma)_{(k_1, \dots, k_n)}$ where $k_i = \#Star(i) - 1$, $i = 1, \dots, n$. If $l = 0$ then from \mathcal{U}_Γ after anti-symmetrization we obtain a pre- L_∞ -morphism.

Let $\gamma_1, \dots, \gamma_n$ be polyvector fields on \mathbb{R}^d of degrees $(k_1+1), \dots, (k_n+1)$, and f_1, \dots, f_m be functions on \mathbb{R}^d . We are going to write a formula for function Φ on \mathbb{R}^n :

$$\Phi := (\mathcal{U}_\Gamma(\gamma_1 \otimes \cdots \otimes \gamma_n))(f_1 \otimes \cdots \otimes f_m) .$$

The formula for Φ is the sum over all configurations of indices running from 1 to d , labeled by E_Γ :

$$\Phi = \sum_{I: E_\Gamma \rightarrow \{1, \dots, d\}} \Phi_I ,$$

where Φ_I is the product over all $n + m$ vertices of Γ of certain partial derivatives of functions g_j and of coefficients of γ_i .

Namely, with each vertex i , $1 \leq i \leq n$ of the first type we associate function ψ_i on \mathbb{R}^d which is a coefficient of the polyvector field γ_i :

$$\psi_i = \langle \gamma_i, dx^{I(e_i^1)} \otimes \cdots \otimes dx^{I(e_i^{k_i+1})} \rangle .$$

Here we use the identification of polyvector fields with skew-symmetric tensor fields as

$$\xi_1 \wedge \cdots \wedge \xi_{k+1} \longrightarrow \sum_{\sigma \in S_{k+1}} sgn(\sigma) \xi_{\sigma_1} \otimes \cdots \otimes \xi_{\sigma_{k+1}} \in \Gamma(\mathbb{R}^d, T^{\otimes(k+1)}) .$$

For each vertex \bar{j} of second type the associated function $\psi_{\bar{j}}$ is defined as f_j .

Now, at each vertex of graph Γ we put a function on \mathbb{R}^d (i.e. ψ_i or $\psi_{\bar{j}}$). Also, on edges of graph Γ there are indices $I(e)$ which label coordinates in \mathbb{R}^d . In the next step we put into each vertex v instead of function ψ_v its partial derivative

$$\left(\prod_{e \in E_\Gamma, e=(*, v)} \partial_{I(e)} \right) \psi_v ,$$

and then take the product over all vertices v of Γ . The result is by definition the summand Φ_I .

Construction of the function Φ from the graph Γ , polyvector fields γ_i and functions f_j , is invariant under the action of the group of affine transformations of \mathbb{R}^d because we contract upper and lower indices.

We define an L_∞ -morphism $\mathcal{U} : T_{poly}^\bullet(\mathbb{R}^d) \rightarrow D_{poly}^\bullet(\mathbb{R}^d)$ by the formula for its n -th derivative \mathcal{U}_n , $n \geq 1$, considered as a skew-symmetric polylinear map

$$\begin{aligned} \mathcal{U}_n : \otimes^n T_{poly}^\bullet(\mathbb{R}^d) &\rightarrow D_{poly}^\bullet(\mathbb{R}^d)[1 - n] : \\ \mathcal{U}_n &= \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} W_\Gamma \times \mathcal{U}_\Gamma \end{aligned}$$

Here $G_{n,m}$ denotes the set of all admissible graphs with n vertices of the first type, m vertices of the second type and $2n + m - 2$ edges, $n \geq 1, m \geq 0$.

Theorem ([K], Sect. 6.4). \mathcal{U} is L_∞ -morphism and also a quasi-isomorphism. L_∞ -morphism \mathcal{U} is equivariant under affine transformations.

The fact that \mathcal{U} is quasi-isomorphism follows directly from the fact that $\mathcal{U}_1 = \varphi_{\text{HKR}}$ and Hochschild–Kostant–Rosenberg Theorem.

Formula (3) applied to the L_∞ -morphism \mathcal{U} defines the Kontsevich universal formula for the deformation quantization.

3. A SKETCH OF THE PROOF OF THEOREM 2 [K]

The condition that \mathcal{U} is an L_∞ -morphism can be written as follows:

$$(4) \quad \begin{aligned} & f_1 \cdot (\mathcal{U}_n(\gamma_1 \wedge \dots \wedge \gamma_n)) (f_2 \otimes \dots \otimes f_m) \pm (\mathcal{U}_n(\gamma_1 \wedge \dots \wedge \gamma_n)) (f_1 \otimes \dots \otimes f_{m-1}) \cdot f_m + \\ & + \sum_{i=1}^{m-1} \pm (\mathcal{U}_n(\gamma_1 \wedge \dots \wedge \gamma_n)) (f_1 \otimes \dots \otimes (f_i \cdot f_{i+1}) \otimes \dots \otimes f_m) + \\ & + \sum_{i \neq j} \pm (\mathcal{U}_{n-1}([\gamma_i, \gamma_j] \wedge \dots \wedge \widehat{\gamma}_i \wedge \dots \wedge \widehat{\gamma}_j \wedge \dots \wedge \gamma_n)) (f_1 \otimes \dots \otimes f_m) + \\ & + \frac{1}{2} \sum_{k, l \geq 1, k+l=n} \frac{1}{k!l!} \sum_{\sigma \in \Sigma_n} \pm [\mathcal{U}_k(\gamma_{\sigma_1} \wedge \dots \wedge \gamma_{\sigma_k}), \mathcal{U}_l(\gamma_{\sigma_{k+1}} \wedge \dots \wedge \gamma_{\sigma_n})] (f_1 \otimes \dots \otimes f_m) = 0. \end{aligned}$$

It is clear that one can write the r.h.s. of (4) as a linear combination

$$(5) \quad \sum_{\Gamma} c_{\Gamma} \cdot \mathcal{U}_{\Gamma}(\gamma_1 \otimes \dots \otimes \gamma_n)(f_1 \otimes \dots \otimes f_m)$$

of expressions \mathcal{U}_{Γ} for admissible graphs Γ with n vertices of the first type, m vertices of the second type, and $2n+m-3$ edges, where $n, m \geq 0$, $2n+m-3 \geq 0$.

The coefficients c_{Γ} are equal to the quadratic-linear combinations of the weights $W_{\Gamma'}$.

We want to check that the coefficient c_{Γ} vanishes for each graph Γ .

The idea is to identify the coefficient c_{Γ} with the integral over the boundary $\partial \overline{C}_{n,m}$ of the closed differential form constructed from Γ as in Sect. 2. The Stokes formula gives the vanishing:

$$(6) \quad \int_{\partial \overline{C}_{n,m}} \bigwedge_{e \in E_{\Gamma}} d\Phi_e = \int_{\overline{C}_{n,m}} d \left(\bigwedge_{e \in E_{\Gamma}} d\Phi_e \right) = 0.$$

The boundary strata of codimension 1 are of the following two types (see [K], Sect. 5):

- (S1): points from subset $S \subset \{1, \dots, n\}$, $\#S \geq 2$ of the first type move close to each other; the corresponding boundary stratum is equal to $\partial_S \overline{C}_{n,m} = \overline{C}_{\#S} \times \overline{C}_{n-\#S+1, m}$
- (S2): points from subset $S \subset \{1, \dots, n\}$ of the first type and points from the subset $S' \subset \{1, \dots, m\}$ of the second type, such that $2\#S + \#S' \geq 2$, $\#S + \#S' \leq n+m-1$, move close to each other and to \mathbb{R} ; the boundary stratum is equal to

$$\partial_{S, S'} \overline{C}_{n,m} = \overline{C}_{\#S, \#S'} \times \overline{C}_{n-\#S, m-\#S'+1}.$$

One have:

$$(7) \quad 0 = \int_{\partial \overline{C}_{n,m}} \bigwedge_{e \in E_{\Gamma}} d\Phi_e = \sum_S \int_{\partial_S \overline{C}_{n,m}} \bigwedge_{e \in E_{\Gamma}} d\Phi_e + \sum_{S, S'} \int_{\partial_{S, S'} \overline{C}_{n,m}} \bigwedge_{e \in E_{\Gamma}} d\Phi_e.$$

The idea is to identity the summands of the last sum with summands of the r.h.s. of (4).

Case S1: the integral (7) vanishes except the case $\#S = 2$ and the two points are connected by an edge \vec{e} (see [K], Sect. 6.6). The case $\#S = 2$ corresponds to the

summands of (4) with the bracket of polyvector fields. The integral in the r.h.s. of (7) is equal, up to 2π , to the integral corresponded to the graph Γ_1 obtained from the graph Γ by the contraction of the edge \vec{e} . Let us note, that the graph Γ_1 has $(n - 1)$ vertices of the first type, m vertices of the second type, and $2n + m - 4 = 2(n - 1) + m - 2$ edges.

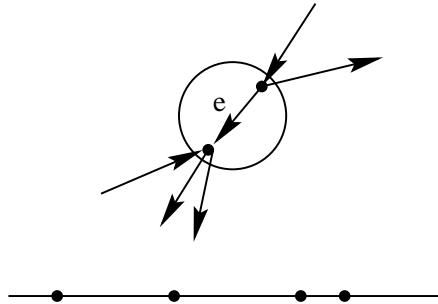


FIGURE 1.

Case 2: One can show ([K], Sect.6.4.2.2) that the integral in (7) vanishes except the case when there does not exist any “external” edge starting in the points of the subset $S \sqcup S'$. The typical situation is shown on Fig. 2.

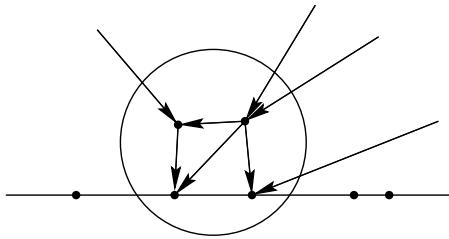


FIGURE 2.

This case is corresponded to the Gerstenhaber bracket of polydifferential operators in the r.h.s. of (4). The integral is equal to the product of the two weights $W_{\Gamma_1} \times W_{\Gamma_2}$.

How to calculate the coefficient c_Γ :

Let Γ be an admissible graph with n vertices of the first type, m vertices of the second type, and $2n + m - 3$ edges. We consider the following two types of representations of the graph Γ :

- (R1): it is a representation of the form $\Gamma = \Gamma' \sqcup \vec{e}$, where the edge \vec{e} connects two vertices of Γ' of the first type (see Fig. 1)
- (R2): it is a representation of the form $\Gamma = \Gamma_1 \cup \Gamma_2$ where: 1) both graphs Γ_2 and $\bar{\Gamma}_1 = \Gamma/\Gamma_2$ (the contraction of Γ_2 to a vertex of the second type) have n_i vertices of the first type, m_i vertices of the second type, and $2n_i + m_i - 2$ edges ($i = 1, 2$); 2) there does not exist any edge starting in the new vertex $= [\Gamma_2]$ of the graph $\bar{\Gamma}_1$. (See Fig. 2).

Any representation of the types (R1), (R2) of the graph Γ has a contribution in the coefficient c_Γ , and c_Γ is the sum over all the possible representations. According to the Stokes formula, the sum of all these contributions is equal to 0. On the other hand, the

contributions of the representations are in 1–1 correspondence with summands in the r.h.s. of (4).

4. WE WANT TO PROVE THE L_∞ -FORMALITY CONJECTURE FOR \mathbb{R}^∞ , OR HOW THE DG LIE ALGEBRA $[T_{\text{poly}}^\bullet]_{\text{lin}}$ APPEARS.

The difficulty in the problem of the extending of the result of Section 2, 3 for the space \mathbb{R}^∞ (in any sense) is the divergence of the polydifferential operators corresponded to the graphs with oriented cycles (between vertices of the first type), as is shown on Fig. 3.

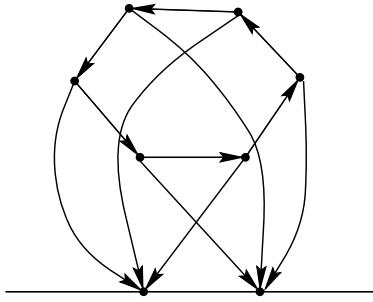


FIGURE 3.

We want to define a new class of “restricted” admissible graphs for the definition of the L_∞ -morphism \mathcal{U} in the Section 2 such that:

- (i) restricted admissible graphs do not contain any oriented cycles;
- (ii) the class of restricted admissible graphs is compatible with the two operations (R1) and (R2) (see Sect. 3), in the sense explained below.

We claim that such a class of restricted admissible graphs does not exist.

4.1. We try to exclude all the graphs with oriented cycles. Let us suppose that the restricted class of admissible graphs contains graphs with non-oriented cycles between vertices of the first type, for example, a graph with a cycle such that all its edges have right orientation except the one unique edge \vec{e} (the general case is the same). Then the representation of the type (R1) $\Gamma = \Gamma' \sqcup \vec{e}$ does not appear in the r.h.s. of (4) because the graph Γ' is not restricted admissible and is not appeared in the definition of the L_∞ -morphism \mathcal{U} . On the other hand, this representation appears in formula (7). Consequently, the summand in (4) and in (7) are not in 1–1 correspondence, and we may not use the arguments of the Stokes formula. Therefore, restricted admissible graphs may not have any (non-oriented) cycle between the vertices of the first type.

4.2. Restricted admissible graphs may not have any cycle (formed by vertices both of the first and the second types). Let us suppose that there exists a restricted admissible graph containing any cycle, like is shown on the Fig. 4.

Then the Gerstenhaber bracket generates a graph with (non-oriented) cycle between vertices of the first type, as is shown in Fig. 5.

The graph shown at the right side of Figure 5 should be restricted admissible, in a contradiction with Sect. 4.1. Then, the restricted admissible graphs may not have any cycle, and it is an easy exercise to prove that the set of these graphs is empty.

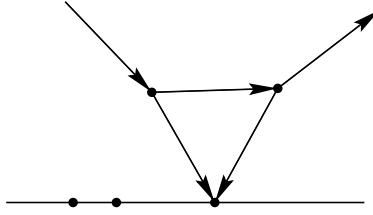


FIGURE 4.

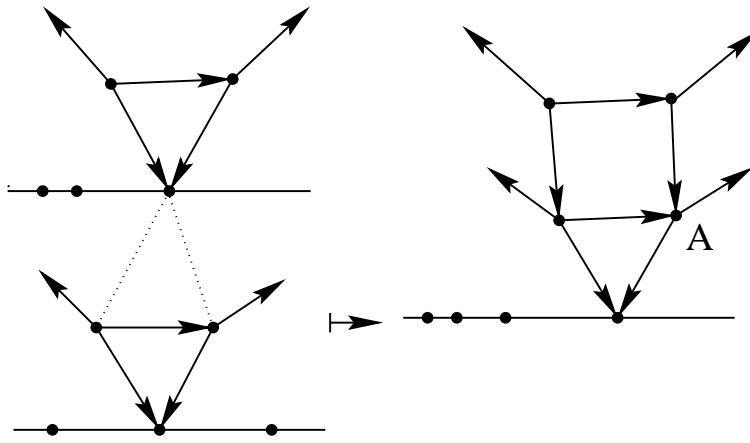


FIGURE 5.

4.3. Differential graded Lie algebra $[T_{\text{poly}}^\bullet]_{\text{lin}}$ of polyvector fields with linear coefficients. The situation described in Sect. 4.2 will not appear for the dg Lie algebra $[T_{\text{poly}}^\bullet]_{\text{lin}}$. Indeed, let us suppose that we consider only *linear* polyvector fields. Then, there exists not more than 1 edge ending at each vertex of the first type. The situation of Sect. 4.2 will not appear because there exist 2 edges ending at the vertex A on the right-hand side of Fig. 5, and the right-hand graph defines zero polydifferential operator.

Let us summarize. Let $G_{n,m}^r$ be the set of admissible graphs (see Definition 2.1) with n vertices of the first type, m vertices of the second type, $2n + m - 2$ edges, and which do not contain any (non-oriented) cycle between the vertices of the first type. The map $\mathcal{U}_n^{\text{lin}}: \otimes^n [T_{\text{poly}}^\bullet]_{\text{lin}} \rightarrow D_{\text{poly}}^\bullet[1-n]$ is defined as follows:

$$(8) \quad \mathcal{U}_n = \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}^r} W_\Gamma \times \mathcal{U}_\Gamma$$

where the weight W_Γ and the polydifferential operator \mathcal{U}_Γ are defined as in Sect. 2. Then formula (8) defines the components of the L_∞ -morphism

$$\mathcal{U}_{\text{lin}}: [T_{\text{poly}}^\bullet]_{\text{lin}} \rightarrow D_{\text{poly}}^\bullet$$

which defines, by formulas (1), (3), a deformation quantization of the Kirillov–Poisson structure on \mathfrak{g}^* (both in finite-dimensional and infinite-dimensional cases). This deformation quantization is exactly the “restricted” Kontsevich’s universal formula, i.e. the

Kontsevich's formula without graphs with any (=oriented in the linear case) cycles between vertices of the first type. According to the theorem of V. Kathotia [Ka], it is exactly the CBH-quantization.

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